

# CONTINUED FRACTION ALGORITHM FOR STURMIAN COLORINGS OF TREES

DONG HAN KIM AND SEONHEE LIM

**ABSTRACT.** Factor complexity  $b_\phi(n)$  for a vertex coloring  $\phi$  of a regular tree is the number of colored  $n$ -balls up to color-preserving automorphisms. Sturmian colorings are colorings of minimal unbounded factor complexity  $b_\phi(n) = n + 2$ .

In this article, we prove an induction algorithm for Sturmian colorings using colored balls in a way analogous to induction algorithm of Sturmian words. Furthermore, we characterize Sturmian colorings in terms of the data for the induction algorithm.

## 1. INTRODUCTION

Factor complexity (or subword complexity or block complexity)  $b_u(n)$  of an infinite word  $u$ , which counts the number of distinct  $n$ -subwords, have been studied for a long time in symbolic dynamics [5]. A classical theorem of Hedlund and Morse says that Sturmian sequences, which are sequences of minimal unbounded complexity  $b_u(n) = n + 1$ , correspond to irrational rotations [6].

Factor complexity was generalized from sequences to vertex colorings of a regular tree in [7], see also [8]: for a vertex coloring  $\phi : VT \rightarrow \mathcal{A}$  of a  $d$ -regular tree  $T$ , the *factor complexity*  $b_\phi(n)$  (called *subword complexity* in [7]) is the cardinality of the set  $\mathbf{B}_\phi(n)$  of classes of colored  $n$ -balls appearing in  $T$  colored by  $\phi$  up to coloring-preserving isomorphisms of  $n$ -balls. Let us denote the tree  $T$  colored by  $\phi$  by  $T_\phi$ .

A coloring  $\phi$  is called *periodic* if  $b_\phi(n)$  is bounded. A coloring  $\phi_g$  associated to an automorphism  $g$  of a uniform tree  $T$  was constructed in [9] and [1] so that  $g$  is a commensurator element of a cocompact lattice  $\Gamma \subset G = \text{Aut}(T)$  if and only if  $\phi_g$  is a periodic coloring. (See [9], Chapter 6 of [2] for commensurators in trees and [3], [12] for general context.) More generally, to any automorphism of a tree, one can associate a vertex coloring, thus classifying vertex colorings of trees may give us a tool to classify automorphisms of a tree.

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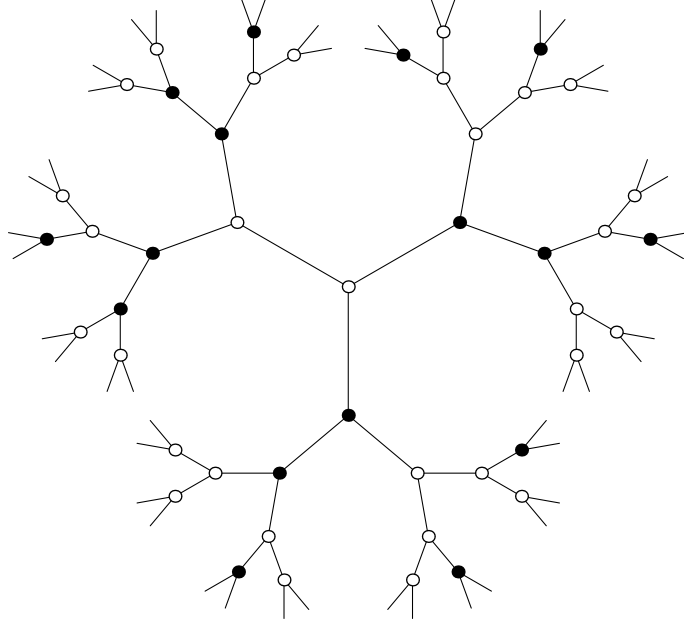


FIGURE 1. An example of Sturmian tree

A coloring  $\phi$  is called *Sturmian* if it has minimal unbounded factor complexity  $b_\phi(n)$ , which is  $n + 2$ . See Figure 1 for an example of a Sturmian coloring. The sets  $\mathbf{B}_\phi(n)$  for the coloring  $\phi$  in Figure 1 (for  $n = 0, 1, 2$ ) are:

$$\begin{aligned} \mathbf{B}_\phi(0) &= \{ \circ, \bullet \}, \\ \mathbf{B}_\phi(1) &= \left\{ \begin{array}{c} \circ \quad \bullet \\ \diagup \quad \diagdown \\ \circ \end{array}, \begin{array}{c} \circ \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}, \begin{array}{c} \circ \quad \bullet \\ \diagup \quad \diagdown \\ \circ \end{array} \right\}, \\ \mathbf{B}_\phi(2) &= \left\{ \begin{array}{c} \bullet \quad \circ \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \circ \quad \bullet \end{array}, \begin{array}{c} \bullet \quad \circ \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \circ \end{array}, \begin{array}{c} \circ \quad \bullet \quad \circ \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \circ \end{array}, \begin{array}{c} \bullet \quad \circ \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \circ \end{array} \right\}. \end{aligned}$$

Let  $G$  be the group of color-preserving automorphisms of  $T$ , which is a subgroup of the automorphism group  $\text{Aut}(T)$ . Let  $X := G \backslash T$  be the graph obtained from quotienting  $T$  by  $G$ .

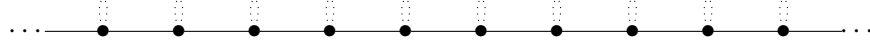
Note that  $G$  is not necessarily a discrete subgroup of  $\text{Aut}(T)$ , but one can often find a subgroup  $\Gamma \subset G$  discrete in  $\text{Aut}(T)$  such that  $X = \Gamma \backslash T$ . It has a structure of edge-indexed graph  $(X, i)$ , i.e. a graph with index  $i : EX \rightarrow \mathbb{N}$  on the set of *oriented* edges. It is in fact the edge-indexed graph associated to the graph of groups  $(X, \text{Stab}_G(\cdot))$  which is the graph  $X$  with stabilizers  $\text{Stab}_G(x)$  attached to each  $x \in VX \cup EX$ . (See Chapter 2 of [2] for details.)

The tree  $T$  is the universal cover of the edge-indexed graph  $(X, i)$  (see Appendix of [2]). The graph  $X$  is again colored by  $\phi$  and the original coloring is the lift of  $\phi$  on  $X$  to  $T$ . In a previous work, we had characterized the quotient graph  $X$  (see Section 6 for definition of bounded type):

**Theorem 1.1** ([7]). *Any Sturmian coloring is a lift of a coloring on a graph  $X$  which is an infinite ray (with loops possibly attached):*



or a biinfinite line (with loops possibly attached):



Moreover, if it is of bounded type, then the graph  $X$  is an infinite ray (with loops possibly attached).

Note that the graph  $X$  for a Sturmian coloring of bounded type can be the edge-indexed graph associated to a lattice of Nagao type (see Chapter 10 of [2]).

In this article, we show an induction algorithm of Sturmian colorings that characterizes Sturmian colorings completely. We show that the coloring  $\phi$  on the quotient graph  $X$  of any Sturmian coloring enjoys an algorithm analogous to an induction algorithm of Sturmian words.

Let us recall the induction algorithm of Sturmian words, which stems from continued fraction algorithm of irrational numbers. For a given infinite word  $u$ , the Rauzy graph  $\mathfrak{G}_n$  is a finite graph whose vertices are distinct  $n$ -words and whose oriented edges are pairs of consecutive  $n$ -words. For Sturmian words, the Rauzy graph is always a union of two cycles with an intersection which is a vertex (case (i)) or a segment (union of edges) (case (ii)). Moreover, one can construct  $\mathfrak{G}_{n+1}$  from  $\mathfrak{G}_n$ , namely  $\mathfrak{G}_{n+1}$  is the line graph of  $\mathfrak{G}_n$  in case (ii), and is the line graph minus one edge in case (i) (see [4] for details).

There is an induction algorithm, namely, two sequences of words  $(A_n)$  and  $(B_n)$  are constructed as follows:  $A_{-1} = 0, B_{-1} = 1$  and

$$(1.1) \quad A_{n+1} = A_n, B_{n+1} = A_n B_n \quad \text{or} \quad A_{n+1} = B_n A_n, B_{n+1} = B_n.$$

Rauzy showed that both sequences  $A_n, B_n$  have the same limit which is a characteristic Sturmian word (i.e. Sturmian word  $x$  such that  $0x, 1x$  are also Sturmian), and conversely any characteristic Sturmian word is the limit of two such sequences [10].

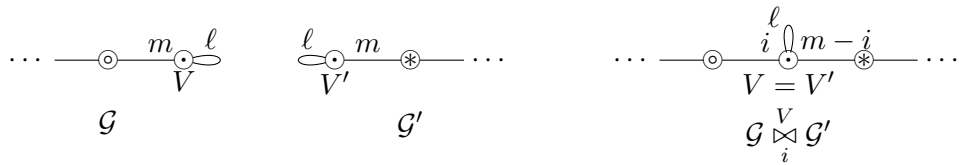
Now let us describe our induction algorithm of Sturmian colorings. For a given Sturmian coloring  $\phi$ , let  $\mathcal{G}_n$  be the graph whose vertices are colored  $n$ -balls in  $T_\phi$  up to color-preserving isomorphisms and whose edges are pairs of colored  $n$ -balls in  $T_\phi$  with adjacent centers. A fundamental feature of Sturmian colorings is that for each  $n$ , there is a unique class of  $n$ -balls, called

the *special  $n$ -ball* and denoted by  $S_n$ , which is a restriction (to the concentric  $n$ -ball) of two distinct classes of  $(n+1)$ -balls, which we denote by  $A_{n+1}, B_{n+1}$ . The graph  $\mathcal{G}_n$  can be thought of as an analogue of Rauzy graphs  $\mathfrak{G}_n$ .

As the graph  $\mathcal{G}_n$  does not have the information of edge indices, we introduce edge-indexed graphs  $\mathcal{G}_n^A, \mathcal{G}_n^B$  whose union is  $\mathcal{G}_n$  as graphs. More precisely, let  $i_A(S_n, Y_n), i_B(S_n, Y_n)$  be the number of balls colored by  $Y_n$  adjacent to  $S_n$  in  $A_{n+1}, B_{n+1}$ , respectively, and let  $i_A(X_n, Y_n) = i_B(X_n, Y_n)$  be the number of  $n$ -balls colored by  $Y_n$  adjacent to  $X_n \neq S_n$ , which does not depend on the position of the ball in  $T$ . Let  $\mathcal{G}_n^A, \mathcal{G}_n^B$  be the edge-indexed graphs defined over the subgraphs of  $\mathcal{G}_n$  consisting of vertices and edges which are connected by a path from the special  $n$ -ball  $S_n$  with edges of positive index for  $i_A, i_B$ , respectively. They are endowed with the edge-index  $i_A, i_B$ , respectively. These edge-index graphs  $\mathcal{G}_n^A, \mathcal{G}_n^B$  appear in the graph  $X$  and they are the building blocks of  $(X, \phi)$ .

**Definition 1.2.** [Sum] Let  $\mathcal{G}, \mathcal{G}'$  be two edge-indexed graphs which have one end colored by an  $n$ -ball  $V$  which is  $C_n \neq S_n$  or  $S_n = C_n$ .

(i) For  $1 \leq i < m$ , the  $(i)$ -sum  $\mathcal{G} \overset{V, V'}{\underset{i}{\bowtie}} \mathcal{G}'$  of  $\mathcal{G}$  and  $\mathcal{G}'$  at  $(V, V')$  is the edge-indexed graph defined as follows : the vertex set is  $V\mathcal{G} \cup V\mathcal{G}'$ , where only the vertices  $V \in \mathcal{G}$  and  $V' \in \mathcal{G}'$  are identified. The oriented edge set is  $EG \cup EG'$ , where the loops at  $V$  in  $\mathcal{G}$  and in  $\mathcal{G}'$  are identified. The edge-index is  $i, m-i$  for the non-loop edges at  $V$  in  $\mathcal{G}, \mathcal{G}'$ , respectively, and  $\ell$  for the loop at  $V$ . Note that  $\ell$  can be zero.



(ii) For  $1 \leq i \leq \ell_1, 1 \leq j \leq \ell_2$ , the  $(i, j)$ -sum  $\mathcal{G} \underset{i, j}{\bowtie} \mathcal{G}'$  of  $\mathcal{G}$  and  $\mathcal{G}'$  at  $(V, V')$  is the edge-indexed graph with vertex set  $V\mathcal{G} \sqcup V\mathcal{G}'$  and the edge set  $EG \sqcup EG' \sqcup e$  with one new edge  $e$  between  $V$  in  $\mathcal{G}$  and  $V$  in  $\mathcal{G}'$ . The edge-index is  $\ell_1 - i, \ell_2 - j$  for loops from  $\mathcal{G}$  and  $\mathcal{G}'$ , respectively, and  $i, j$  for  $e$ .



By carefully analyzing adjacencies between special  $n$ -balls, restriction of special  $(n+1)$ -balls and extensions of the special  $(n-1)$ -ball to  $n$ -balls, we derive the following characterization, an analog of induction algorithm described in (1.1).

**Theorem 1.3.** *Let  $\phi$  be a Sturmian coloring without a cycle in  $\mathcal{G}_n$ , for all sufficiently large  $n$ . There exists  $K \in [0, \infty]$  and a sequence  $(n_k, v_k)_{k \geq 0}$  such that  $n_k = k$  for  $0 \leq k \leq K$  and*

$$\begin{aligned} \mathcal{G}_n^A &\cong \mathcal{G}_{n-1}^A, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie_{v_k} \mathcal{G}_{n-1}^B, & \text{if } 0 \leq n < n_K, \\ \mathcal{G}_n^A &\cong \mathcal{G}_{n-1}^A \bowtie_{v_k} \mathcal{G}_{n-1}^B, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie_{v'_k} \mathcal{G}_{n-1}^B \quad \text{or} \quad \mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie_{v_k} \mathcal{G}_{n-1}^B, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B, & \text{if } n = n_K, \\ \mathcal{G}_n^A &\cong \mathcal{G}_{n-1}^A, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B & \text{if } n \neq n_K, k \geq K, \\ \mathcal{G}_n^A &\cong \mathcal{G}_{n-1}^A \bowtie_{v_k} \mathcal{G}_{n-1}^B, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B \quad \text{or} \quad \mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie_{v_k} \mathcal{G}_{n-1}^B, & \text{if } n = n_K, k \geq K. \end{aligned}$$

Here,  $v_k, v'_k$  are edge index data. (See Theorem 3.4 for details.)

Let  $(\alpha_k)$  be the sequence of letters  $A, B$  such that  $\mathcal{G}_{n_k}^{\alpha_k}$  is larger than the other and  $\alpha_K = A$ . The sequence  $(\alpha_k, v_k)$  in the above theorem has a role corresponding to the slope for the irrational rotation associated to a Sturmian word (or the ratio of alphabets  $a$  and  $b$  appearing in the word). The freedom coming from intercept (starting point of the irrational rotation) of Sturmian words are replaced by a sequence  $\beta_k$  satisfying Lemma 4.6.

We also characterize the sequences  $(\alpha_k, v_k)$  that appear in the above theorem and call them admissible sequences (see Definition 4.1):  $\alpha_k$  can be freely chosen and the condition for  $v_k$  comes from the fact that the universal cover of the resulting edge-indexed graph should have degree  $d$ .

There is an inverse process of the above theorem as follows. For such an admissible sequence  $(\alpha_k, v_k)$  and a choice of  $\beta_k$ , we show that the direct limit of  $\mathcal{G}_{n_k}^{\beta_k}$  is the quotient graph  $(X, \phi)$  of a Sturmian coloring  $\phi$ . Note that  $n_k$  are determined by  $(\alpha_k, v_k)$ . If  $\phi$  is of bounded type, the sequence  $\beta_k$  has a further restriction: they are eventually the same letter.

Furthermore, if we start with a Sturmian coloring  $\phi$ , for a suitable choice of  $\beta_k$ , we recover the coloring  $\phi$ . See Theorem 4.7, Proposition 5.4 and Theorem 6.1 for details.

**Theorem 1.4.** *Let  $\phi$  be a Sturmian coloring.*

- (1) *If  $\phi$  is without a cycle in  $\mathcal{G}_n$ , for all sufficiently large  $n$ , then there is an admissible sequence of pairs  $\{(\alpha_k, v_k)\}_{k=0}^\infty$  and a sequence  $\beta_k$  such that the direct limit of  $\mathcal{G}_{n_k}^{\beta_k}$  is the original Sturmian coloring  $(X, \phi)$ .*
- (2) *If  $\phi$  is of bounded type, then  $(X, \phi) = \varinjlim \mathcal{G}_n^*$  where  $*$  =  $A$  or  $B$ .*
- (3) *If  $\mathcal{G}_n$  has a cycle for some  $n$ , then  $\phi$  is of bounded type.*

If  $\phi$  is of bounded type, then by part (2) of the above theorem,  $(X, \phi)$  is the limit of  $\mathcal{G}_n^A$ 's or  $\mathcal{G}_n^B$ 's, say  $\mathcal{G}_n^A$ 's. We show that  $\mathcal{G}_n^B$  also plays a role in the sense that the coloring is a countable union of periodic colorings, where the periodic coloring is determined by  $\mathcal{G}_n^B$ . See Theorem 6.2 for details.

The article is organized as follows. In Section 2, we gather preliminary facts about colored balls and define the graph  $\mathcal{G}_n$  and the edge-indexed graphs  $\mathcal{G}_n^A, \mathcal{G}_n^B$ . In Section 3, we study colored balls of acyclic colorings, i.e. colorings for which  $\mathcal{G}_n$  does not have a cycle for all  $n$ . In Section 4, we show the induction algorithm for acyclic colorings and characterize them in terms of  $(\alpha_k, v_k)$  and  $\beta_k$ . In Section 5, cyclic Sturmian colorings are treated. We investigate Sturmian colorings of bounded type further in Section 6.

## 2. GRAPHS OF COLORED BALLS

Let  $T$  be a  $d$ -regular tree, i.e. the degree of each vertex is  $d$ . Let  $VT, ET$  be the set of vertices and the set of oriented edges of  $T$ , respectively. The group  $G$  of automorphisms of  $T$  is a locally compact topological group with compact-open topology. Consider the path metric  $d$  on  $T$  with edge length all equal to 1. An  $n$ -ball around  $x$  is defined by  $\mathcal{B}_n(x) = \{y \in VT \cup ET : d(x, y) \leq n\}$ .

Throughout the paper, a coloring  $\phi : VT \rightarrow \mathcal{A}$  will be Sturmian, i.e., a coloring of subball complexity  $b(n) = n + 2$ . Since  $b(0) = 2$ ,  $\mathcal{A}$  has two elements. Set  $\mathcal{A} = \{a, b\}$ .

**2.1. Preliminary : basic properties of Sturmian colorings.** In this subsection, we recall preliminary facts from [7] and prove basic properties of Sturmian colorings.

**Definition 2.1.** For a coloring  $\phi$ , denote the colored tree by  $T_\phi$ .

- (1) Two vertices  $x$  and  $y$  are said to be *in the same class* if there exists a color-preserving automorphism of  $T_\phi$  sending  $x$  to  $y$ .
- (2) Two balls  $\mathcal{B}_n(x), \mathcal{B}_n(y)$  are called *equivalent* if there exists a color-preserving isomorphism between them. Such an equivalence class is called a *colored  $n$ -ball* and is denoted it by  $[\mathcal{B}_n(x)]$ . A colored  $n$ -ball is called *admissible* if it appears in  $T_\phi$ .
- (3) Let  $\mathbf{B}_\phi(n)$  be the set of admissible colored  $n$ -balls. The *subball complexity*  $b_\phi(n)$  of  $\phi$  is defined by  $b_\phi(n) = b(n) = |\mathbf{B}_\phi(n)|$ .

**Definition 2.2.** Let  $\phi$  be a Sturmian coloring, or more generally a coloring of subword complexity  $b(n) = n + k$  for some  $k \in \mathbb{N}$ .

- (1) Since  $b(n+1) - b(n) = 1$ , for each  $n$ , there is a unique colored  $n$ -ball which is contained concentrically in two distinct  $(n+1)$ -balls. We call it the *special (colored)  $n$ -ball* and denote it by  $S_n$  for  $n \geq 0$ .
- (2) Denote by  $A_n, B_n$  the two distinct colored  $n$ -balls containing  $S_{n-1}$  concentrically, for  $n \geq 1$ .
- (3) The central colored  $n$ -ball of the special  $(n+1)$ -ball  $S_{n+1}$  will be denoted by  $C_n$ .

One of the most basic properties of a Sturmian coloring is that if an  $n$ -ball  $B$  is not special, then there is a unique colored  $(n+1)$ -ball, denoted by  $\overline{B}$  and called an extension of  $B$ , containing  $B$  concentrically. For the notational simplicity, we denote the empty ball by  $S_{-1} = A_{-1} = C_{-1}$ . Note that  $B_{-1}$  is not defined.

We will call a colored  $n$ -ball an  $n$ -ball when there is no ambiguity, in particular for  $A_n, B_n, C_n$  and  $S_n$ .

**Lemma 2.3.** *Without loss of generality, we assume that  $S_0 = A_0 = [a]$  and  $B_0 = [b]$ .*

- (1) *We can choose  $\{A_n\}, \{B_n\}$  so that  $A_{n+1}, B_{n+1}$  is adjacent to  $A_n, B_n$ , respectively.*
- (2) *For each colored  $n$ -ball  $D$ , there are at most two distinct colored  $n$ -balls adjacent to  $D$  apart from  $D$  itself.*
- (3) *If  $A_n \neq S_n$  (respectively  $B_n \neq S_n$ ), then each  $n$ -ball colored by  $A_n$  (respectively  $B_n$ ) is adjacent to an  $S_n$ .*
- (4) *Each  $n$ -ball colored by  $S_n$  is adjacent to an  $n$ -ball colored by  $C_n$ .*

*Proof.* For (1) and (2), see Lemma 2.11, Lemma 3.3 and Lemma 3.8 in [7]. For (3), let  $x, z \in VT$  be the centers of a ball colored by  $A_n, A_{n+1}$ , respectively. By part (1), there exists  $w \in VT$  adjacent to  $z$  which is the center of  $A_n$ . Since  $A_n$  is not special, there exists a color preserving  $f$  from the unique  $\overline{A_n}$  around  $w$  to the  $\overline{A_n}$  around  $x$ . Let  $y = f(z)$ , then  $d(x, y) = 1$  and by  $f$  the  $n$ -ball around  $y$  is isomorphic to  $S_n$  which is the  $n$ -ball around  $z$ . For (4), let  $x \in VT$  be the center of a ball colored by  $S_n$  and say  $A_{n+1}$ . If  $A_{n+1} \neq S_{n+1}$ , then by part (3) there exists a neighboring vertex  $y \in VT$  which is a center of  $S_{n+1}$ , i.e., a center of  $C_n$ . If  $A_{n+1} = S_{n+1}$ , then part (1) implies that  $x$  is adjacent to the center  $y$  of  $A_{n+2}$ , which is the center of  $S_{n+1}$  and  $C_n$ .  $\square$

Note that so far, the choice of  $\{A_n\}, \{B_n\}$  may not be unique.

**Lemma 2.4.** *Suppose an  $n$ -ball colored by  $S_n$  is adjacent to a colored  $n$ -ball  $D$  which is not equals to any of  $A_n, B_n, S_n$ . Then any  $n$ -ball colored by  $S_n$  is adjacent to  $D$ . Furthermore, if  $D$  is distinct from  $C_n$ , then  $S_n \neq C_n$ .*

*Proof.* Let  $x, y \in VT$  be adjacent vertices which are centers of  $n$ -balls colored by  $S_n, D$ , respectively. Let  $x'$  be the center of another  $n$ -ball colored by  $S_n$  and  $f$  be the color preserving isomorphism from  $n$ -ball of  $x$  to  $n$ -ball of  $x'$ . Then  $y' = f(y)$  is the center of  $\underline{D}$  with  $d(x', y') = 1$ , thus the centre of  $D$ . Therefore, every ball colored by  $S_n$  is adjacent to a ball colored by  $D$ .

Let  $D \neq C_n$ . Suppose that  $S_n = C_n$ . Then either  $A_{n+1} = S_{n+1}$  or  $B_{n+1} = S_{n+1}$ . By Lemma 2.3 (3), it follows that  $B_{n+1}(\neq S_{n+1})$  is adjacent to  $S_{n+1} = A_{n+1}$  or  $A_{n+1}(\neq S_{n+1})$  is adjacent to  $S_{n+1} = B_{n+1}$ . Therefore, there exist  $A_{n+1}, B_{n+1}$  which are adjacent to each other. Since each  $A_{n+1}, B_{n+1}$  are adjacent to  $A_n, B_n$  and to  $D$  by Lemma 2.3 (1), by the first assertion of the lemma, there exist  $A_{n+1}$  and  $B_{n+1}$  which are adjacent to  $B_{n+1}, \overline{A_n}, \overline{D}$  and to  $A_{n+1}, \overline{B_n}, \overline{D}$  respectively. Since either  $A_n \neq S_n$  or  $B_n \neq S_n$ , either  $B_{n+1}, \overline{A_n}, \overline{D}$  or  $A_{n+1}, \overline{B_n}, \overline{D}$  are distinct, which contradicts Lemma 2.3 (2).  $\square$

**2.2. Graph of colored balls.** Recall that the vertices of  $X$  is the color-preserving isomorphism classes of vertices of the colored tree  $T_\phi$ , where two vertices  $t_1$  and of  $t_2$  of the tree are in the same class if the colored  $n$ -balls of  $t_1, t_2$  are isomorphic as colored balls, for any  $n \in \mathbb{N}$ .

**Definition 2.5.** We define the graph  $\mathcal{G}_n$  as follows. The set of vertices  $V\mathcal{G}_n = \mathcal{B}_\phi(n)$  are colored  $n$ -balls of  $T_\phi$ . Two  $n$ -balls are connected by an edge if there are corresponding  $n$ -balls in  $T$  which are adjacent.

**Lemma 2.6.** (1) If  $S_n$  is adjacent to an  $n$ -ball  $D$  which is distinct from  $A_n, B_n, C_n, S_n$ , then  $D$  is a vertex of degree 1 in  $\mathcal{G}_n$  and there is a cycle in  $\mathcal{G}_{n+1}$ .  
 (2) If  $B_{n+1}$  (resp.  $A_{n+1}$ ) is adjacent to  $A_n$  (resp.  $B_n$ ) which is distinct from  $S_n, C_n$ , then there is a cycle in  $\mathcal{G}_{n+1}$ .

*Proof.* (1) By Lemma 2.4,  $S_n \neq C_n$ . Since  $S_n \neq D$ , both  $A_{n+1}$  and  $B_{n+1}$  are distinct from  $S_{n+1}$  and  $\overline{D}$ . Lemma 2.3 (3) implies that both  $A_{n+1}$  and  $B_{n+1}$  are adjacent to  $S_{n+1}$ . By Lemma 2.4, any  $S_n$  is adjacent to  $D$ , thus both  $A_{n+1}$  and  $B_{n+1}$  are adjacent to  $\overline{D}$ . Since  $\overline{D} \neq S_{n+1}$ ,  $\overline{D}$  extends uniquely to  $\overline{\overline{D}}$ , i.e., both  $A_{n+1}$  and  $B_{n+1}$  are adjacent to  $\overline{\overline{D}}$ . Therefore, the path with vertices  $[S_{n+1}A_{n+1}\overline{\overline{D}}B_{n+1}S_{n+1}]$  is a cycle in  $\mathcal{G}_{n+1}$ .

(2) Since  $A_n \neq C_n$ , by Lemma 2.3 (1) and by assumption, the nonspecial  $(n+1)$ -ball  $\overline{A_n}$  is adjacent to  $A_{n+1}$  and  $B_{n+1}$ . Note that since  $A_n \neq S_n$ ,  $\overline{A_n}$  differs from both  $A_{n+1}$  and  $B_{n+1}$ . If  $S_n \neq C_n$ , then  $S_{n+1}$  differs from both  $A_{n+1}$  and  $B_{n+1}$ . Thus, by Lemma 2.3 (3), the path with vertices  $[S_{n+1}A_{n+1}\overline{A_n}B_{n+1}S_{n+1}]$  is a cycle in  $\mathcal{G}_{n+1}$ . If  $S_n = C_n$ , then  $S_{n+1}$  is one of  $A_{n+1}, B_{n+1}$ . By Lemma 2.3 (3), the path  $[S_{n+1}\overline{A_n}B_{n+1}S_{n+1}]$  or  $[S_{n+1}\overline{A_n}A_{n+1}S_{n+1}]$  is a cycle which belongs to  $\mathcal{G}_{n+1}$ .  $\square$



**Lemma 2.7.** *There exists  $K \in [0, \infty]$  such that  $S_n = A_n = C_n$  if and only if  $0 \leq n < K$ .*

- (1) *There exists  $K \in [-1, \infty]$  such that  $S_n = A_n = C_n$  if and only if  $-1 \leq n \leq K$ .*
- (2) *If  $A_{n+1} = S_{n+1}$  and  $B_{n+1} = C_{n+1}$ , then  $S_n = A_n = C_n$ .*

*Proof.* Suppose that  $S_{n+1} = A_{n+1} = C_{n+1}$  and  $n \geq 0$ . Then by Lemma 2.4,  $S_{n+1}$  is adjacent to only  $B_{n+1}$  and itself. By Lemma 2.3 (2),  $B_{n+1}$  has exactly one more adjacent vertex  $D$  since  $\mathcal{G}_{n+1}$  is connected and  $|V\mathcal{G}_{n+1}| \geq 3$ . Then  $S_n = \underline{B_{n+1}} = \underline{A_{n+1}} = \underline{S_{n+1}} = C_n$  and  $S_n$  is adjacent to only  $\underline{D}$  and itself. By Lemma 2.3 (1), there exist  $n$ -balls colored by  $S_n$  adjacent to  $A_n$  and  $B_n$  respectively. Thus,  $A_n = S_n = C_n$  and  $B_n = \underline{D}$ . (The case  $B_n = S_n = C_n$  and  $A_n = \underline{D}$  does not occur since otherwise  $A_{n+1}$  would be adjacent to  $\underline{D}$  which is a contradiction to the fact that  $S_{n+1}$  is adjacent to only  $B_{n+1}$  and itself.) If  $S_0 = A_0 \neq C_0$ , then put  $K = 0$ .  $\square$

Note that by our choice of  $S_0 = A_0$ , the case  $B_n = S_n = C_n$  does not occur.

**2.3. Edge-indexed Graphs  $\mathcal{G}_n^A, \mathcal{G}_n^B$  of colored balls.** Although the vertices of  $\mathcal{G}_n$  are equivalence classes of vertices of  $X$ ,  $\mathcal{G}_n$  do not resemble the graph  $X$  even locally, because of the special ball  $S_n$ , which can be extended to two ways in  $T_\phi$ . In this section, we define two edge-indexed graphs  $\mathcal{G}_n^A, \mathcal{G}_n^B$  so that  $X$  locally looks like either  $\mathcal{G}_n^A$  or  $\mathcal{G}_n^B$ . Since  $\phi$  is Sturmian, there are exactly two distinct colored  $(n+1)$ -balls  $A_{n+1}, B_{n+1}$  extending  $S_n$ .

**Definition 2.8.** Define the indices  $i, i_A, i_B : \bigcup_n (V\mathcal{G}_n \times V\mathcal{G}_n) \rightarrow \mathbb{N}$  as follows.

- (1) If an  $n$ -ball  $X_n$  is not special, let  $i(X_n, Y_n)$  be the number of  $n$ -balls colored by  $Y_n$  adjacent to  $X_n$ , which is independent of the position of  $X_n$  in the tree.
- (2) Define  $i_A(S_n, Y_n), i_B(S_n, Y_n)$  to be the number of  $Y_n$  adjacent to  $S_n$  in  $A_{n+1}, B_{n+1}$ , respectively. For simplicity, for  $U \neq S_n$  denote  $i_A(U, V) = i_B(U, V) = i(U, V)$ .

In particular,  $i(X_n, Y_n) = 0$  if there is no edge between  $X_n$  and  $Y_n$ . It is clear that the set of vertices  $V\mathcal{G}_n^A \cup V\mathcal{G}_n^B$  is the set of colored  $n$ -balls  $\mathbf{B}_\phi(n)$ .

**Definition 2.9.** Define  $\mathcal{G}_n^A (\mathcal{G}_n^B)$  to be the edge-indexed oriented graph whose vertices are those which are connected by a path from  $S_n$  with edges of positive index for  $i_A$  ( $i_B$ , respectively). Their oriented edge set is the set of oriented edges between vertices in  $\mathcal{G}_n^A (\mathcal{G}_n^B, respectively)$  with positive index for  $i_A$  ( $i_B$ , respectively), endowed with the index  $i_A$  ( $i_B$ , respectively).

For  $n = -1$ , define  $\mathcal{G}_{-1}^A = \mathcal{G}_{-1}^B$  to be the edge-indexed graph consisting of the vertex of empty ball  $S_{-1} = C_{-1}$  and a loop on it indexed by  $d$ , the degree of  $T$ .

Note that the 1-neighborhood of a given vertex in  $\mathcal{G}_n^A$  and  $\mathcal{G}_n^B$  as graphs colored by  $\phi_n$  are identical except at  $S_n$ , since any non-special  $n$ -balls has a unique extension to  $(n+1)$ -ball.

The following lemma is immediate by definition. The index  $i_B$  enjoys similar properties as  $i_A$ .

**Lemma 2.10.** *Let  $V \neq S_n$ . Then for each  $n \geq -1$*

(1) *For an  $n$ -ball  $U \neq S_n, C_n$  we have*

$$i(U, V) = i(\overline{U}, \overline{V}), \quad i(U, S_n) = i(\overline{U}, A_{n+1}) + i(\overline{U}, B_{n+1}).$$

(2) *If  $C_n \neq S_n$ , then*

$$i(C_n, V) = i_A(S_{n+1}, \overline{V}), \quad i(C_n, S_n) = i_A(S_{n+1}, A_{n+1}) + i_A(S_{n+1}, B_{n+1}),$$

$$i_A(S_n, V) = i(A_{n+1}, \overline{V}), \quad i_A(S_n, S_n) = i(A_{n+1}, A_{n+1}) + i(A_{n+1}, B_{n+1}).$$

*Similar properties hold for  $i_B$ .*

(3) *If  $C_n = S_n$ , say  $S_{n+1} = A_{n+1}$ , then*

$$i_A(S_n, V) = i_A(A_{n+1}, \overline{V}) \quad i_A(S_n, S_n) = i_A(A_{n+1}, A_{n+1}) + i_A(A_{n+1}, B_{n+1})$$

$$= i_B(A_{n+1}, \overline{V}) \quad = i_B(A_{n+1}, A_{n+1}) + i_B(A_{n+1}, B_{n+1})$$

$$i_B(S_n, V) = i(B_{n+1}, \overline{V}) \quad i_B(S_n, S_n) = i(B_{n+1}, A_{n+1}) + i(B_{n+1}, B_{n+1}).$$

**Definition 2.11.** We say that a Sturmian coloring is cyclic if there is a cycle in  $\mathcal{G}_n$  for some  $n$ .

We say that a Sturmian coloring is acyclic if it is not cyclic.

**Example 2.12.** For the coloring  $\phi$  in Figure 1, we have the sequence of  $\mathcal{G}_n$  as follows.

$$\begin{aligned} \mathcal{G}_0^A : & \quad \begin{array}{c} 2 \quad 1 \\ \circ \text{---} \bullet \\ S_0 \quad B_0 \end{array} \\ \mathcal{G}_0 : & \quad \begin{array}{c} \circ \text{---} \bullet \\ S_0 \quad B_0 \end{array} \\ A_0 = S_0 = \circ, \quad B_0 = C_0 = \bullet; & \quad \mathcal{G}_0^B : \begin{array}{c} 1 \quad 2 \quad 2 \quad 1 \\ \circ \text{---} \bullet \\ S_0 \quad B_0 \end{array} \\ \mathcal{G}_1^A : & \quad \begin{array}{c} 2 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1 \\ \circ \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ A_1 \quad S_1 \quad B_1 \end{array} \\ \mathcal{G}_1 : & \quad \begin{array}{c} \circ \text{---} \bullet \text{---} \circ \\ A_1 \quad S_1 \quad B_1 \end{array} \\ A_1 = \begin{array}{c} \circ \quad \bullet \\ \diagup \quad \diagdown \\ \circ \end{array}, \quad B_1 = C_1 = \begin{array}{c} \circ \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \quad S_1 = \overline{B_0} = \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \circ \end{array}; & \quad \mathcal{G}_1^B : \begin{array}{c} 1 \quad 2 \quad 2 \quad 1 \\ \bullet \text{---} \bullet \text{---} \bullet \\ S_1 \quad B_1 \end{array} \\ \overline{A_1} = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \\ \circ \end{array}, \quad \overline{B_1} = S_2 = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \\ \circ \end{array}, \quad A_2 = \begin{array}{c} \circ \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \\ \bullet \end{array}, \quad B_2 = \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \\ \bullet \end{array}. \end{aligned}$$

$$\begin{aligned}
\mathcal{G}_2^A : & \quad \begin{array}{c} \text{2} \quad \text{1} \quad \text{1} \quad \text{1} \quad \text{2} \quad \text{1} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{A}_1 \quad \text{A}_2 \quad \text{S}_2 \end{array} \\
\mathcal{G}_2 : & \quad \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \\ \text{A}_1 \quad \text{A}_2 \quad \text{S}_2 \\ \text{B}_2 \end{array} \\
\mathcal{G}_2^B : & \quad \begin{array}{c} \text{1} \quad \text{2} \quad \text{2} \quad \text{1} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{B}_2 \quad \text{S}_2 \end{array}
\end{aligned}$$

Note that  $S_2 = C_2$  and  $A_3 = S_3$ .

$$\begin{aligned}
\mathcal{G}_3^A : & \quad \begin{array}{c} \text{1} \quad \text{2} \quad \text{1} \quad \text{1} \quad \text{1} \quad \text{2} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{A}_3 \quad \text{C}_3 \quad \text{A}_1 \end{array} \\
\mathcal{G}_3 : & \quad \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{B}_2 \quad \text{B}_3 \quad \text{A}_3 \quad \text{C}_3 \quad \text{A}_1 \end{array} \\
\mathcal{G}_3^B : & \quad \begin{array}{c} \text{1} \quad \text{2} \quad \text{2} \quad \text{1} \quad \text{1} \quad \text{2} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{B}_2 \quad \text{B}_3 \quad \text{A}_3 \quad \text{C}_3 \quad \text{A}_1 \end{array} \\
\mathcal{G}_4^A : & \quad \begin{array}{c} \text{1} \quad \text{2} \quad \text{1} \quad \text{1} \quad \text{1} \quad \text{2} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{A}_4 \quad \text{S}_4 \quad \text{C}_4 \end{array} \\
\mathcal{G}_4 : & \quad \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{B}_2 \quad \text{B}_3 \quad \text{A}_4 \quad \text{S}_4 \quad \text{C}_4 \\ \text{B}_4 \end{array} \\
\mathcal{G}_4^B : & \quad \begin{array}{c} \text{1} \quad \text{2} \quad \text{2} \quad \text{1} \quad \text{1} \quad \text{2} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{B}_2 \quad \text{B}_3 \quad \text{B}_4 \quad \text{S}_4 \quad \text{C}_4 \end{array} \\
\mathcal{G}_5^A : & \quad \begin{array}{c} \text{1} \quad \text{2} \quad \text{1} \quad \text{1} \quad \text{1} \quad \text{2} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{A}_4 \quad \text{A}_5 \quad \text{S}_5 \end{array} \\
\mathcal{G}_5 : & \quad \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{B}_2 \quad \text{B}_3 \quad \text{A}_4 \quad \text{A}_5 \quad \text{S}_5 \\ \text{B}_4 \quad \text{B}_5 \end{array} \\
\mathcal{G}_5^B : & \quad \begin{array}{c} \text{1} \quad \text{2} \quad \text{2} \quad \text{1} \quad \text{1} \quad \text{2} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{B}_2 \quad \text{B}_3 \quad \text{B}_4 \quad \text{B}_5 \quad \text{S}_5 \end{array}
\end{aligned}$$

### 3. ACYCLIC STURMIAN COLORINGS

Throughout Section 3 and Section 4, we assume that  $\phi$  is acyclic, i.e.  $\mathcal{G}_n$  does not have any cycle for all  $n$ . In the process of defining  $\{A_n, B_n\}$ , we impose the condition that if both of  $A_{n+1}, B_{n+1}$  are adjacent to both  $A_n, B_n$ , the number of  $B_n$  adjacent to  $A_{n+1}$  is at least the number of  $A_n$  adjacent to  $A_{n+1}$ . It follows that  $\{A_n\}, \{B_n\}$  are unique.

**Lemma 3.1.** *The following properties hold. (Similar properties hold for  $i_B$ .)*

- If furthermore  $A_{n+1} \neq C_{n+1}$ , then  $i(C_n, S_n) = i_B(S_{n+1}, B_{n+1})$ .

- (ii)  $C_n \neq S_n$  implies that  $S_{n+1} \neq A_{n+1}$ ,  $S_{n+1} \neq B_{n+1}$ . If  $A_{n+1}$  and  $B_{n+1}$  are adjacent, then  $[S_{n+1}A_{n+1}B_{n+1}S_{n+1}]$  is a cycle in  $\mathcal{G}_{n+1}$ , thus  $i(A_{n+1}, B_{n+1}) = i(B_{n+1}, A_{n+1}) = 0$ . Moreover, if  $A_{n+1} \neq C_{n+1}$ , then by Lemma 2.6 (ii) we deduce that  $i_B(S_{n+1}, A_{n+1}) = 0$ . By Lemma 2.10, the second assertion follows.

- Definition 3.2.** [Sum] Let  $\mathcal{G}, \mathcal{G}'$  be two edge-indexed graphs which have one end colored by an  $n$ -ball  $V$  which is  $C_n \neq S_n$  or  $S_n = C_n$ .

- $$\begin{array}{ccc}
\dots \text{---} \circ \text{---} m \overset{\ell}{\bullet} & \overset{\ell}{\bullet} \text{---} m \text{---} * \text{---} \dots & \dots \text{---} \circ \text{---} \overset{\ell}{i} \cap \overset{m-i}{*} \text{---} \dots \\
V & V' & V = V' \\
\mathcal{G} & \mathcal{G}' & \mathcal{G} \underset{i}{\vee}_V \mathcal{G}'
\end{array}$$

- (ii) For  $1 \leq i \leq \ell_1, 1 \leq j \leq \ell_2$ , the  $(i, j)$ -sum  $\mathcal{G} \boxtimes_{i,j} \mathcal{G}'$  of  $\mathcal{G}$  and  $\mathcal{G}'$  at  $(V, V')$  is the edge-indexed graph with vertex set  $V\mathcal{G} \sqcup V\mathcal{G}'$  (the vertices  $V$  in  $\mathcal{G}$  and  $\mathcal{G}'$  are distinct in the sum) and the edge set  $E\mathcal{G} \sqcup E\mathcal{G}' \sqcup e$  with one new edge  $e$  between  $V$  in  $\mathcal{G}$  and  $V$  in  $\mathcal{G}'$ . The edge-index is  $\ell_1 - i, \ell_2 - j$  for loops from  $\mathcal{G}$  and  $\mathcal{G}'$ , respectively, and  $i, j$  for  $e$ .



Let us call  $V$  ( $V, V'$ ) the *joining vertex* (vertices) of the  $(i)$ -sum ( $(i, j)$ -sum, respectively). For simplicity, let us denote  $\mathcal{G}'$  by  $\mathcal{G} \bowtie_{0,j} \mathcal{G}'$ . We will denote by  $\mathcal{G} \bowtie \mathcal{G}'$  either an  $i$ -sum or an  $(i, j)$ -sum of  $\mathcal{G}$  and  $\mathcal{G}'$ . In this section, when we omit the vertex  $V, V'$ , all the  $(i)$ -sum and  $(i, j)$ -sum of  $\mathcal{G}_n^A$ ,  $\mathcal{G}_n^B$  are at  $(C_n, C_n)$ .

**Theorem 3.3.** *Sturmian colorings enjoy the following properties.*

- (1) If  $A_{n+1} \neq S_{n+1}$  and  $A_{n+1} \neq C_{n+1}$ , then  $\mathcal{G}_{n+1}^B \cong \mathcal{G}_n^B$ .
- (2) If  $S_n \neq C_n$  and  $A_{n+1} = C_{n+1}$ , then for some  $i$ ,  $\mathcal{G}_{n+1}^B \cong \mathcal{G}_n^A \bowtie_i \mathcal{G}_n^B$ .
- (3) If  $A_{n+1} = S_{n+1}$  or if  $S_n = C_n$  and  $A_{n+1} = C_{n+1}$ , then for some  $i, j$ ,  $\mathcal{G}_{n+1}^B \cong \mathcal{G}_n^A \bowtie_{i,j} \mathcal{G}_n^B$ .

By symmetry, similar properties hold for  $\mathcal{G}_{n+1}^A$ .

*Proof.* (1) If  $S_n \neq C_n$ , then by Lemma 3.1 (ii),  $i(C_n, S_n) = i_B(S_{n+1}, B_{n+1})$  and  $i_B(S_n, S_n) = i(B_{n+1}, B_{n+1})$ . Also, by Lemma 3.1 (i),  $i(B_n, S_n) = i(\overline{B_n}, B_{n+1})$  if  $B_n \neq S_n, C_n$ . Since there is no other adjacent  $n$ -ball to  $S_n$ , we get  $\mathcal{G}_{n+1}^B \cong \mathcal{G}_n^B$ .

If  $S_n = C_n$ , then  $B_{n+1} = S_{n+1}$  and  $A_{n+1} \neq C_{n+1}$ . Thus by Lemma 3.1 (iii)  $i_B(S_n, S_n) = i_B(B_{n+1}, B_{n+1})$ . Also, by Lemma 3.1 (i),  $i(B_n, S_n) = i(\overline{B_n}, B_{n+1})$  if  $B_n \neq S_n = C_n$ . Since there is no other adjacent  $n$ -ball to  $S_n$ , we get  $\mathcal{G}_{n+1}^B \cong \mathcal{G}_n^B$ .

(2) If  $S_n \neq C_n$  and  $A_{n+1} = C_{n+1}$ , then by Lemma 3.1 (ii)  $i_B(S_n, S_n) = i(B_{n+1}, B_{n+1})$  and by Lemma 2.10 (2),

$$i(C_n, S_n) = i_B(S_{n+1}, A_{n+1}) + i_B(S_{n+1}, B_{n+1}),$$

where  $i_B(S_{n+1}, A_{n+1}) > 0$ ,  $i_B(S_{n+1}, B_{n+1}) > 0$  from Lemma 2.3 (4) and (1), respectively. This case corresponds to

$$\mathcal{G}_{n+1}^B \cong \mathcal{G}_n^A \bowtie_i \mathcal{G}_n^B,$$

where  $i = i_B(S_{n+1}, A_{n+1})$ .

(3) If  $A_{n+1} = S_{n+1}$  (thus  $S_n = C_n$ ), by Lemma 2.10 (3), we have

$$i_A(S_n, S_n) = i_B(A_{n+1}, A_{n+1}) + i_B(A_{n+1}, B_{n+1}),$$

$$i_B(S_n, S_n) = i(B_{n+1}, A_{n+1}) + i(B_{n+1}, B_{n+1}).$$

where  $i_B(A_{n+1}, B_{n+1}) > 0$ ,  $i(B_{n+1}, A_{n+1}) > 0$  by Lemma 2.3 (1). This case corresponds to

$$\mathcal{G}_{n+1}^B \cong \mathcal{G}_n^A \bowtie_{i,j} \mathcal{G}_n^B,$$

where  $i = i_B(A_{n+1}, B_{n+1})$  and  $j = i(B_{n+1}, A_{n+1})$ .

Suppose that  $A_{n+1} \neq S_{n+1}$ ,  $S_n = C_n$  and  $A_{n+1} = C_{n+1}$ . By Lemma 2.10 (3) we have

$$\begin{aligned} i_A(S_n, S_n) &= i(A_{n+1}, A_{n+1}) + i(A_{n+1}, B_{n+1}), \\ i_B(S_n, S_n) &= i_B(B_{n+1}, A_{n+1}) + i_B(B_{n+1}, B_{n+1}). \end{aligned}$$

where  $i(A_{n+1}, B_{n+1}) > 0$ ,  $i_B(B_{n+1}, A_{n+1}) > 0$  by Lemma 2.3 (1) and (4) respectively. This case corresponds to

$$\mathcal{G}_{n+1}^B \cong \mathcal{G}_n^A \bowtie_{i,j} \mathcal{G}_n^B,$$

where

$$i = i(A_{n+1}, B_{n+1}), \quad j = i_B(B_{n+1}, A_{n+1}). \quad \square$$

Using the above observations, we have the following theorem:

**Theorem 3.4.** *Let  $\phi$  be an acyclic Sturmian coloring. There exist  $K \in [0, \infty]$  and a sequence  $(n_k)_{k=0}^\infty$  with the following property:*

(1) *For  $0 \leq k < K$ , we have  $A_k = C_k = S_k$ , thus*

$$\mathcal{G}_k^A \cong \mathcal{G}_{k-1}^A, \quad \mathcal{G}_k^B \cong \mathcal{G}_{k-1}^A \bowtie_{i,j} \mathcal{G}_{k-1}^B.$$

(2) *For  $k = K$ , we have the following cases:*

(a) *If  $A_k = S_k$ ,  $B_k = C_k$ , then for some  $i \neq i'$*

$$\mathcal{G}_k^A \cong \mathcal{G}_{k-1}^A \bowtie_{i',j} \mathcal{G}_{k-1}^B, \quad \mathcal{G}_k^B \cong \mathcal{G}_{k-1}^A \bowtie_{i,j} \mathcal{G}_{k-1}^B.$$

(b) *If  $B_k = S_k$ , then  $C_k = \overline{B_{k-1}}$  and*

$$\mathcal{G}_k^A \cong \mathcal{G}_{k-1}^A \bowtie_{i,j} \mathcal{G}_{k-1}^B, \quad \mathcal{G}_k^B = \mathcal{G}_{k-1}^B.$$

(3) *For every  $n > K$ ,*

$$\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B, \quad \text{if } n \neq n_k,$$

$$\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B \quad \text{or} \quad \mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A, \quad \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, \quad \text{if } n = n_k.$$

*Proof.* (1) For  $0 \leq n < K$ , by Lemma 2.7,  $A_n = C_n = S_n \neq B_n$ . Theorem 3.3 (1) and (3) implies that  $\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A$  and  $\mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie_{i,j} \mathcal{G}_{n-1}^B$  for some  $i, j$ .

(2) Let  $n = K$ . Then  $C_{n-1} = S_{n-1}$  which implies either (a)  $S_n = A_n$  or (b)  $S_n = B_n$ .

(a)  $S_n = A_n$  : we have  $C_n \neq A_n = S_n$  by the definition of  $K$ . By Lemma 2.3 (4) and Lemma 2.6 (1) we have  $B_n = C_n$ . Thus, Theorem 3.3 (3) implies that  $\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie_{i',j} \mathcal{G}_{n-1}^B$  and  $\mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie_{i,j} \mathcal{G}_{n-1}^B$  for some  $i', i, j$ .

(b)  $S_n = B_n \neq A_n$ : by the choice of  $A_0$  and  $B_0$ ,  $n = K \geq 1$ .  $S_n$  is adjacent to distinct balls  $A_n$  and  $\overline{B_{n-1}}$ . By Lemma 2.6 (1), we have  $C_n = \overline{B_{n-1}} \neq A_n$ , thus Theorem 3.3 (1) and (3) implies that  $\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie_{i,j} \mathcal{G}_{n-1}^B$  for some  $i, j$  and  $\mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B$ .

(3) Assume that  $n > K$ . If  $S_{n-1} = C_{n-1}$ , then we have either  $A_n = S_n$  or  $B_n = S_n$ . Suppose that  $B_n \neq A_n = S_n$ . If  $B_n = C_n$ , then by Lemma 2.3 (4) and Lemma 2.6 (1),  $A_n$  is adjacent to  $A_n$  and  $B_n$  only. By Lemma 2.3 (1),  $A_n$  is adjacent to  $A_{n-1}$ , which implies that  $A_{n-1} = S_{n-1} = C_{n-1}$ , which contradicts  $n \geq K$ . Thus  $B_n \neq C_n$  and Theorem 3.3 (1) and (3) implies that  $\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A$  and  $\mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie_{i,j} \mathcal{G}_{n-1}^B$  for some  $i, j$ . For the case  $A_n \neq B_n = S_n$ , we apply the same argument.

If  $S_{n-1} \neq C_{n-1}$ , then we have  $A_n, B_n \neq S_n$ . Suppose that  $B_n \neq A_n = C_n$  or  $A_n \neq B_n = C_n$ . Then by Theorem 3.3 (1), (2) for some  $i$ ,  $\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A$ ,  $\mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie_i \mathcal{G}_{n-1}^B$  or  $\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie_i \mathcal{G}_{n-1}^B$ ,  $\mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B$ . In case of  $A_n, B_n \neq C_n$ , by Theorem 3.3 (1),  $\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A$  and  $\mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B$ . This is the case of  $n \neq n_k$ .  $\square$

For an alphabet  $\alpha$ , denote by  $\overline{\alpha}$  the alphabet different from  $\alpha$ .

**Definition 3.5.** (1) Define  $\alpha_k$  so that

$$|V\mathcal{G}_{n_k}^{\alpha_k}| > |V\mathcal{G}_{n_k}^{\overline{\alpha_k}}|,$$

which is well-defined if  $k \neq K$ . Set  $\alpha_K = A$ .

(2) For cases appearing in Theorem 3.4, define  $v_k$  by

$$v_k = \begin{cases} (i, j), & \text{for case (1),} \\ (i, i', j), & \text{for case (2)(a),} \\ (0, i, j), & \text{for case (2)(b),} \\ (i) \text{ or } (i, j), & \text{for case (3).} \end{cases}$$

**Lemma 3.6.** *The special ball  $S_{n_k}$  is a vertex of degree 1 in  $\mathcal{G}_{n_k}^{\alpha_k}$ . Put  $m = |V\mathcal{G}_{n_k}^{\alpha_k}|$ . For  $i = 0, 1, \dots, m-1$ , the vertex of  $\mathcal{G}_{n_k}^{\alpha_k}$  of distance  $i$  from  $S_{n_k}$ , which we denote by  $[i]$ , is the central  $n_k$ -ball of  $S_{n_k+i}$ .*

Thus,

$$n_{k+1} = \begin{cases} n_k + m - 1, & \text{if there is a } (i)\text{-sum at } n_{k+1}, \\ n_k + m, & \text{if there is a } (i, j)\text{-sum at } n_{k+1}. \end{cases}$$

Moreover, none of the vertices of  $\mathcal{G}_{n_k-1}^{\alpha_k}$  is the center of a special  $n$ -ball if  $n_k + 1 \leq n \leq n_{k+1}$  for  $(i)$ -sum at  $n_k$  and if  $n_k \leq n \leq n_{k+1}$  for  $(i, j)$ -sum at  $n_k$ .

*Proof.* We may assume that  $\alpha_k = A$ . From the definition of sum, it is immediate that  $C_{n_k-1}$  is a vertex of degree 1 in  $\mathcal{G}_{n_k}^B$ . By Theorem 3.4,  $S_{n_k} \neq C_{n_k}$ . By Lemma 2.3 (4), the only adjacent vertex of  $[0] = S_{n_k}$  is  $[1]$ , which equals to  $C_{n_k}$ . By the canonical projection from  $\mathcal{G}_{n_k}^B$  to  $\mathcal{G}_{n_k+1}^B$ , the vertices  $[0], [1]$  are mapped to  $B_{n_k+1}, S_{n_k+1}$  respectively, thus again Lemma 2.3 (4) implies that  $[2]$  is mapped to  $C_{n_k+1}$ . Inductively, for  $1 \leq i \leq m-1$  the vertex  $[i]$  is mapped to  $S_{n_k+i}$  by the canonical projection from  $\mathcal{G}_{n_k}^B$  to  $\mathcal{G}_{n_k+i}^B$ . By Theorem 3.4, we continue this procedure until  $n_{k+1}$ .  $\square$

**Example 3.7.** Let  $s_i, t_i, i = 1, 2, 3$  be integers satisfying  $t_i \geq 1, s_i \geq 0, s_i + 2t_i = d$  for each  $i = 1, 2, 3$  and  $s_1 \neq s_2$ . Example 7 in [7] is the following:

$$\begin{aligned} Y : & \quad \cdots \quad \overset{c}{\bullet} \quad \overset{d}{\bullet} \quad \overset{c}{\bullet} \quad \overset{c}{\bullet} \quad \overset{d}{\bullet} \quad \overset{c}{\bullet} \quad \overset{d}{\bullet} \quad \cdots \\ X : & \quad \cdots \quad \overset{s_1}{\circ} \overset{s_3}{\circ} \overset{s_2}{\circ} \overset{s_3}{\circ} \overset{s_1}{\circ} \overset{s_3}{\circ} \overset{s_1}{\circ} \overset{s_3}{\circ} \overset{s_2}{\circ} \overset{s_3}{\circ} \overset{s_1}{\circ} \overset{s_3}{\circ} \overset{s_2}{\circ} \cdots \\ & \quad \mathcal{G}_0^A : \overset{s_1 2t_1}{\circ} \overset{2t_3 s_3}{\bullet} \quad \mathcal{G}_0^B : \overset{s_2 2t_2}{\circ} \overset{2t_3 s_3}{\bullet} \\ & \quad \mathcal{G}_1^A : \overset{s_1 2t_1}{\circ} \overset{2t_3 s_3}{\bullet} \quad \mathcal{G}_1^B : \overset{s_1 2t_1}{\circ} \overset{t_3 t_3}{\bullet} \overset{2t_2 s_2}{\circ} \\ & \quad \mathcal{G}_2^A : \overset{s_3 2t_3}{\bullet} \overset{t_1 t_1}{\circ} \overset{t_3 t_3}{\bullet} \overset{2t_2 s_2}{\circ} \quad \mathcal{G}_2^B : \overset{s_1 2t_1}{\circ} \overset{t_3 t_3}{\bullet} \overset{2t_2 s_2}{\circ} \\ & \quad \mathcal{G}_3^A : \overset{s_3 2t_3}{\bullet} \overset{t_1 t_1}{\circ} \overset{t_3 t_3}{\bullet} \overset{2t_2 s_2}{\circ} \quad \mathcal{G}_3^B : \overset{s_1 2t_1}{\circ} \overset{t_3 t_3}{\bullet} \overset{2t_2 s_2}{\circ} \\ & \quad \mathcal{G}_4^A : \overset{s_3 2t_3}{\bullet} \overset{t_1 t_1}{\circ} \overset{t_3 t_3}{\bullet} \overset{2t_2 s_2}{\circ} \quad \mathcal{G}_4^B : \overset{s_1 2t_1}{\circ} \overset{t_3 t_3}{\bullet} \overset{t_2 t_2}{\circ} \overset{t_3 t_3}{\bullet} \overset{t_1 t_1}{\circ} \overset{2t_3 s_3}{\bullet} \end{aligned}$$

In this example,  $K = 0$ , and moreover, we have  $n_k = f_{k+2} - 1$  where  $f_k$  is the Fibonacci sequence defined by  $f_k = f_{k-1} + f_{k-2}$ ,  $f_1 = 1, f_2 = 1$ , i.e.,  $n_0 = 0, n_1 = 1, n_2 = 2, n_3 = 4$  and so on. Note that  $\alpha_k = A$  if  $k$  is even and  $\alpha_k = B$  if  $k$  is odd. For the sequence of  $v_k$  we have  $v_0 = (2t_1, 2t_2, 2t_3)$  and  $v_{3k+i} = (t_{2+i})$  for each  $i = -1, 0, 1$ .



## 4. INDUCTION ALGORITHM FOR ACYCLIC STURMIAN COLORING

In this section, we characterize acyclic Sturmian colorings completely by showing that the converse of Theorem 3.4 also holds : we define admissible sequences which determine sequences of edge-indexed graphs  $\mathcal{F}_k^A, \mathcal{F}_k^B$  constructed by  $(i)$ -sums or  $(i, j)$ -sums recursively, so that the appropriate direct limit  $\mathcal{F}$  is a linear graph canonically colored by  $a, b$ . We show that these colorings are Sturmian.

**4.1. Admissible sequences.** Let us first define the admissible sequences of sum vectors which are associated to acyclic Sturmian colorings.

For  $k > K$ , let us denote by  $e_k^C$  the index of the edge from the common end vertex to its adjacent vertex and by  $e_k^A$  ( $e_k^B$ ) the index of the edge from the non-common end vertex of  $\mathcal{G}_{n_k}^A$  ( $\mathcal{G}_{n_k}^B$ , respectively) to its adjacent vertex.

Let  $\{(\alpha_k, v_k)\}_{k=0}^\infty$  be a sequence of pairs of  $\alpha_k \in \{A, B\}$  and  $v_k \in \mathcal{D} \cup \mathcal{D}^2 \cup \overline{\mathcal{D}} \times \mathcal{D}^2$ , where  $d$  is the degree of the tree and  $\mathcal{D} = \{1, 2, \dots, d-1, d\}$ ,  $\overline{\mathcal{D}} = \{0, 1, 2, \dots, d-1, d\}$ . Put  $K = \min\{k \geq 0 : \alpha_k = A\}$  and  $i_{-1} = 0$ .

**Definition 4.1** (Admissible sequence). We call  $(\alpha_k, v_k)$  an *admissible sequence* if it satisfies the following conditions.

(i) If  $0 \leq k < K$ , then  $v_k = (i_k, j_k) \in \mathcal{D}^2$  satisfies

$$1 \leq i_k < d, \quad 1 \leq j_k \leq d - i_{k-1}.$$

(ii) If  $k = K$ , then  $v_k = (i_k, i'_k, j_k) \in \overline{\mathcal{D}} \times \mathcal{D}^2$  satisfies

$$0 \leq i_k < i'_k \leq d, \quad 1 \leq j_k \leq d - i_{k-1},$$

and furthermore  $i_k > 0$  if  $K = 0$ .

(iii) If  $k > K$ , then

$$\begin{aligned} 1 \leq i_k < e_{k-1}^C, & \quad \text{if } v_k = i_k, \\ 1 \leq i_k \leq d - e_{k-1}^C, \quad 1 \leq j_k \leq d - e_k^C, & \quad \text{if } v_k = (i_k, j_k), \end{aligned}$$

where the sequence  $(e_k^A, e_k^B, e_k^C)_{k \geq K}$  of three indices of end vertices are recursively defined by

$$\begin{aligned} e_K^A = i'_K, \quad e_K^B = i_{K-1}, \quad e_K^C = j_0 & \quad \text{for } i_K = 0, \\ e_K^A = i_K, \quad e_K^B = i'_K, \quad e_K^C = j_0 & \quad \text{for } i_K > 0 \end{aligned}$$

and for  $k > K$

$$e_k^C = e_{k-1}^{\hat{\alpha}_k}, \quad e_k^{\hat{\alpha}_k} = e_{k-1}^C.$$

**Definition 4.2** (Definition of edge-index graphs  $\mathcal{F}_k^A, \mathcal{F}_k^B$ ). To an admissible sequence  $(\alpha_k, v_k)$ , we associate the edge-indexed graphs  $\mathcal{F}_n^A, \mathcal{F}_n^B$  defined as follows.

- (1) For  $k = -1$ ,  $\mathcal{F}_{-1}^A, \mathcal{F}_{-1}^B$  are both the graph with one vertex and one loop of index  $d$ .

$$\begin{array}{c} d \\ \circlearrowleft \\ \bullet \\ \mathcal{F}_{-1}^A, \quad \mathcal{F}_{-1}^B \end{array}$$

- (2) For  $k = 0, \dots, K-1$ , define

$$\mathcal{F}_k^A = \mathcal{F}_{k-1}^A, \quad \mathcal{F}_k^B = \mathcal{F}_{k-1}^A \underset{i_k, j_k}{\overset{V, V'}{\bowtie}} \mathcal{F}_{k-1}^B,$$

where  $V$  is the unique vertex of  $\mathcal{F}_{k-1}^A$ , and the vertex  $V'$  is the unique vertex of  $\mathcal{F}_{k-1}^B$  coming from  $\mathcal{F}_{k-2}^A$  for  $k \geq 1$  and the unique vertex of  $\mathcal{F}_{-1}^B$  for  $k = 0$ .

$$\begin{array}{ccc} \begin{array}{c} d \\ \circlearrowleft \\ \bullet \\ V \\ \mathcal{F}_k^A \end{array} & \begin{array}{c} d - i_k \quad i_k \quad j_k \\ \circlearrowleft \quad \bullet \quad \bullet \\ V' \\ \mathcal{F}_k^B \end{array} & \dots \end{array}$$

- (3) For  $k = K$ , define

$$\mathcal{F}_K^A = \mathcal{F}_{K-1}^A \underset{i'_k, j_k}{\overset{V, V'}{\bowtie}} \mathcal{F}_{K-1}^B, \quad \mathcal{F}_K^B = \mathcal{F}_{K-1}^A \underset{i_k, j_k}{\overset{V, V'}{\bowtie}} \mathcal{F}_{K-1}^B,$$

where  $V, V'$  are as in part (2). According to our definition, if  $i_K = 0$ , then  $\mathcal{F}_K^B = \mathcal{F}_{K-1}^B$ .

- (4) For  $k > K$ , define

$$\mathcal{F}_k^{\alpha_k} = \mathcal{F}_{k-1}^A \underset{v_k}{\bowtie} \mathcal{F}_{k-1}^B, \quad \mathcal{F}_k^{\bar{\alpha}_k} = \mathcal{F}_{k-1}^{\bar{\alpha}_k}.$$

**4.2. Direct limit.** For a given sequence of pairs  $\{\mathcal{F}_k^A, \mathcal{F}_k^B\}$  associated to  $(\alpha_k, v_k)$ , choose a sequence  $\beta_k \in \{A, B\}$  such that

$$(4.1) \quad \beta_k = \alpha_k \text{ or } \beta_{k-1}.$$

We have a natural graph inclusions  $\mathcal{F}_k^{\beta_k} \subset \mathcal{F}_{k+1}^{\beta_{k+1}}$ . Consider the graph  $Y$  which is the direct limit determined by inclusions  $f_{k, k+1} : \mathcal{F}_k^{\beta_k} \hookrightarrow \mathcal{F}_{k+1}^{\beta_{k+1}}$ . The graph  $Y$  has a canonical coloring  $\varphi$ : the color of each vertex  $y$  is  $a$  or  $b$ , according to whether  $y$  comes from  $\mathcal{F}_{-1}^A$  or  $\mathcal{F}_{-1}^B$ , respectively. With the edge index induced from  $\mathcal{F}_k^{\beta_k}$ , let us denote by  $(Y, \varphi)$  the edge-indexed graph  $Y$  colored by  $\varphi$ . For simplicity, let us denote

$$(Y, \varphi) = \varinjlim \mathcal{F}_k^{\beta_k}.$$

By construction, the universal cover of the edge-indexed graph  $Y$  is a  $d$ -regular tree  $T$ , thus  $(Y, \varphi)$  defines a coloring on the tree  $T$ , which is denoted again by  $(Y, \varphi)$ . Note that the sequences  $\beta_k, \beta'_k$  are eventually equal if and only if they have the same direct limit.

The edge-indexed graph  $\mathcal{F}_k^A, \mathcal{F}_k^B$  colored by  $\varphi$  have a  $d$ -regular colored tree as their universal cover, which we denote by  $(T, \varphi_k^A), (T, \varphi_k^B)$ , respectively. Let

$$n_k = |V\mathcal{F}_k^{\alpha_k}| - 2 \quad \text{and} \quad m_k = |V\mathcal{F}_k^{\bar{\alpha}_k}|.$$

For  $n \leq n_k$  denote by  $\mathcal{B}_n(t)$  the colored  $n$ -ball around a lift of  $t \in V\mathcal{F}_k^A(V\mathcal{F}_k^B)$  in  $(T, \varphi_k^A)$  ( $(T, \varphi_k^B)$  respectively).

**Proposition 4.3.** (1) For  $t, t' \in V\mathcal{F}_k^{\alpha_k}$ , we have

$$\mathcal{B}_{n_k}(t) \neq \mathcal{B}_{n_k}(t').$$

Thus  $b_\phi(n_k) \geq n_k + 2$ .

(2) For  $t \in V\mathcal{F}_k^A, t' \in V\mathcal{F}_k^B$  with the same inverse image under  $f_{k-1,k}$ , we have  $\mathcal{B}_{n_k}(t) = \mathcal{B}_{n_k}(t')$ . Thus  $b_\phi(n_k) \leq n_k + 2$ .

(3) The direct limit  $(Y, \varphi)$  is a Sturmian coloring.

*Proof.* For part (1), we use induction. The case  $k = K$  is clear, since the graph  $\mathcal{G}_{n_K}^A$  is colored by  $a$  except one vertex at one end, which is colored by  $b$ .

Suppose  $\mathcal{B}_{n_k}(t) \neq \mathcal{B}_{n_k}(t')$ . Let us denote the vertices of  $\mathcal{F}_k^{\bar{\alpha}_k}$  by  $\mathbf{t}_1, \dots, \mathbf{t}_{m_k}$  starting from the joining vertex for  $\mathcal{F}_k^{\alpha_k}$ . If  $\mathcal{F}_{k+1}^{\alpha_{k+1}}$  is  $(i, j)$ -sum, we claim that for  $l = 1, \dots, m_k$ ,  $\mathcal{B}_{n_k+l}(\mathbf{t}_l) \neq \mathcal{B}_{n_k+l}(\mathbf{t}'_l)$ , where  $\mathbf{t}'_l \in \mathcal{F}_k^{\alpha_k}$ ,  $\mathbf{t}_l$  have the same image under the projection in  $\mathcal{F}_{k-1}^{\bar{\alpha}_k}$ . For  $l = 1$ ,  $\mathbf{t}'_1$  has neighboring vertices from  $\mathcal{F}_k^{\alpha_k}$  where as  $\mathbf{t}_1$  does not. Thus for  $l = 2, \dots, m_k$ ,  $\mathcal{B}_{n_k+l}(\mathbf{t}_l) \neq \mathcal{B}_{n_k+l}(\mathbf{t}'_l)$ , since  $\mathcal{B}_{n_k+l}(\mathbf{t}_l), \mathcal{B}_{n_k+l}(\mathbf{t}'_l)$  contains  $\mathcal{B}_{n_k}(\mathbf{t}_1), \mathcal{B}_{n_k}(\mathbf{t}'_1)$ , respectively, which are distinct. The case  $l = m_k$  corresponding to  $n_{k+1}$ . Similarly, if  $\mathcal{F}_{k+1}^{\alpha_{k+1}}$  is  $(i)$ -sum, the claim holds for  $l = 1, \dots, m_k - 1$ .

For part (2), we use induction again. Suppose that the statement is true for  $n_k$  and let us show it for  $n_{k+1}$ . To show that  $\mathcal{B}_{n_{k+1}}(t) = \mathcal{B}_{n_{k+1}}(t')$ , it is enough to show that the  $m_k$ -balls centered at  $t, t'$  for  $(i, j)$ -sum (and  $(m_k - 1)$ -balls for  $(i)$ -sum, respectively) colored by  $n_k$  balls are equal. If  $\alpha_{k+1} = \alpha_k$ , let us denote the vertices of  $\mathcal{F}_k^{\alpha_k}$  by  $\mathbf{t}_1, \dots, \mathbf{t}_{m_k}$  starting from the joining vertex for  $\mathcal{F}_k^{\alpha_k}$ . Then the vertices  $m_k$ -balls around  $\mathbf{t}_l$  are all coming from vertices, say  $v_i$ 's, of  $\mathcal{F}_k^B$  and  $v'_i$ 's of  $\mathcal{F}_k^A$  which have common projection with  $v_i$  of  $\mathcal{F}_k^B$ . Since each such couple have the same  $n_k$  balls by induction hypothesis, the  $m_k$ -balls around  $\mathbf{t}_l$  colored by  $n_k$ -balls are equal. For the case  $\alpha_{k+1} \neq \alpha_k$ : denote the vertices of  $\mathcal{F}_k^{\bar{\alpha}_k}$  by  $\mathbf{t}_1, \dots, \mathbf{t}_{m_k}$ . For  $l = 1, \dots, m_k$ , the statement for balls centered at  $\mathbf{t}_l$  follows similarly. For  $l = m_k + 1, \dots, n_k$ , the  $m_k$  (or  $m_k - 1$ )-ball centered at  $\mathbf{t}_l$  are contained in  $\mathcal{F}_k^{\bar{\alpha}_k}$ .

For part (3), by part (1) and (2), we have  $b(n_k) = n_k + 2$  for a sequence  $n_k \rightarrow \infty$ , thus  $b(n) = n + 2$  for all  $n$  by [7].  $\square$

**Remark 4.4.** As mentioned in the introduction, Sturmian words correspond to irrational rotations. By considering rotations as maps from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , one can consider “slope” and “intercept” of Sturmian sequences. The colorings have the same factors if and only if the sequences  $(\alpha_k, v_k)$  are the same. In this regard, the sequences  $(\alpha_k, v_k)$  correspond to partial quotients of the slope of a Sturmian sequence and the sequence  $\beta_k$  correspond to the intercept of a Sturmian sequence.

**4.3. Sturmian coloring and direct limits.** For a given Sturmian coloring  $(T, \phi)$ , construct  $\mathcal{G}_n^A, \mathcal{G}_n^B$  as in Section 3, which determine  $n_k$  and  $(\alpha_k, v_k)$  which is clearly admissible. With  $\mathcal{F}_k^A = \mathcal{G}_{n_k}^A, \mathcal{F}_k^B = \mathcal{G}_{n_k}^B$ , choose  $\beta_k$  as follows: for a fixed vertex  $t \in VT$ , choose  $\beta_k \in \{A, B\}$  as follows:

- (i) if  $\mathcal{B}_{n_{k+1}}(t) \in V\overline{\mathcal{F}_{k+1}^{\alpha_{k+1}}}$  then set  $\beta_k = \overline{\alpha}_{k+1}$ .
- (ii) if  $\mathcal{B}_{n_{k+1}}(t) \in V\overline{\mathcal{F}_{k+1}^{\alpha_{k+1}}} - V\overline{\mathcal{F}_{k+1}^{\alpha_{k+1}}}$  then set  $\beta_k = \alpha_{k+1}$ .

If we choose another vertex  $t' \in VT$ , then the sequence  $\beta'_k$  is eventually equal to  $\beta_k$  by the following lemma, thus the direct limit does not depend on the choice of  $t$ .

**Lemma 4.5.** *For any two vertices  $t, t'$ , there exists  $k_0$  such that for all  $k \geq k_0$ ,  $\mathcal{B}_{n_k}(t'') \in \mathcal{G}_{n_k}^{\beta_k}$ , for every  $t''$  between  $t$  and  $t'$  in  $X$ .*

*Proof.* Suppose  $t'$  is on the left hand side of  $t$ . Suppose that there is no such  $k_0$ , and let  $t_0$  be the leftmost vertex in  $\mathcal{G}_{n_k}^{\beta_k}$  for sufficiently large  $k$ . Now let  $t_1, t_2$  be the vertices adjacent to  $t_0$ . We claim that  $t_1, t_2$  are in the same class. If not, then for sufficiently large  $k$ , we have  $\mathcal{B}_{n_k}(t_1) \neq \mathcal{B}_{n_k}(t_2)$ . For sufficiently large  $k$ ,  $\mathcal{B}_{n_k}(t_0)$  has to be the leftmost vertex of  $\mathcal{G}_{n_k}^{\beta_k}$ , and by Proposition 4.3, there is only one colored  $(n_k)$ -ball adjacent to  $t_0$ , which is a contradiction.  $\square$

**Lemma 4.6.** *The sequence  $\beta_k$  defined above satisfies  $\beta_k = \alpha_k$  or  $\beta_{k-1}$ .*

*Proof.* Suppose that  $\beta_k \neq \beta_{k-1} = \alpha_k$ . It corresponds to case (ii) above, thus  $\mathcal{B}_{n_k}(t) \in V\overline{\mathcal{F}_k^{\alpha_k}} - V\overline{\mathcal{F}_k^{\alpha_k}}$ .

If  $\alpha_{k+1} = \alpha_k \neq \beta_k$ , then it falls into the case (i) thus  $\mathcal{B}_{n_{k+1}}(t) \in V\overline{\mathcal{F}_{k+1}^{\alpha_{k+1}}}$ . By definition of  $\alpha_k$ , we have  $\overline{\mathcal{F}_{k+1}^{\alpha_{k+1}}} \simeq \overline{\mathcal{F}_k^{\alpha_{k+1}}} = \overline{\mathcal{F}_k^{\alpha_k}}$ , thus  $\mathcal{B}_{n_k}(t) \in V\overline{\mathcal{F}_k^{\alpha_k}}$ , which is a contradiction.

If  $\alpha_{k+1} = \beta_k \neq \alpha_k$ , then it falls into the case (ii), thus  $\mathcal{B}_{n_{k+1}}(t) \in V\overline{\mathcal{F}_{k+1}^{\alpha_{k+1}}} - V\overline{\mathcal{F}_{k+1}^{\alpha_{k+1}}}$ . By definition  $\overline{\mathcal{F}_{k+1}^{\alpha_{k+1}}} = \overline{\mathcal{F}_k^{\alpha_{k+1}}} \bowtie \overline{\mathcal{F}_k^{\alpha_{k+1}}}$ , thus  $\mathcal{B}_{n_{k+1}}(t) \in V\overline{\mathcal{F}_k^{\alpha_{k+1}}}$ , which is a contradiction since  $\alpha_{k+1} \neq \alpha_k$ .  $\square$

**Theorem 4.7.** *For given  $(X, \phi)$ , let  $(Y, \varphi) = \varinjlim \mathcal{F}_k^{\beta_k}$  be the direct limit as in the previous section. We have*

$$(Y, \varphi) = (X, \phi).$$

*Conversely, if  $(a_k, v_k)$  is an admissible sequence and  $\beta_k$  is any sequence satisfying  $\beta_k = \alpha_k$  or  $\beta_{k-1}$ , then the sequence induced from the coloring  $(Y, \varphi)$  is  $(\alpha_k, v_k)$ .*

*Proof.* By Lemma 4.5, for any vertex  $t \in X$ ,  $\mathcal{B}_{n_k}(t) \in \mathcal{G}_{n_k}^{\beta_k}$  for sufficiently large  $n_k$ . Thus there is an injection  $t \mapsto \varinjlim \mathcal{B}_{n_k}(t)$ , where  $\varinjlim \mathcal{B}_{n_k}(t)$  is the point in the direct limit corresponding to the sequence  $\mathcal{B}_{n_k}(t)$ .  $\square$

## 5. CYCLIC STURMIAN COLORINGS

In this section, we investigate cyclic Sturmian colorings.

**Lemma 5.1.** *Suppose that  $\mathcal{G}_n$  has a cycle. Then*

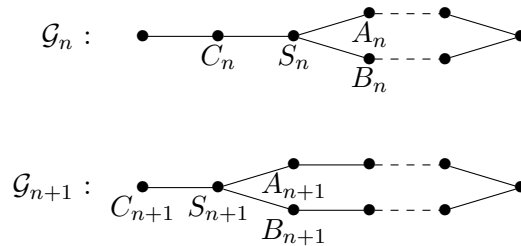
- (i) *the special ball  $S_n$  is in the cycle.*
- (ii) *the special ball  $S_n$  is adjacent to  $A_n, B_n, C_n$  only.*

*Proof.* (i) Suppose that  $\mathcal{G}_n$  has a vertex  $v$  which is not in the cycle. Since  $\mathcal{G}_n$  is connected, there exists a vertex  $w$  in the cycle connected to  $v$ . Since  $S_n$  is the unique vertex of  $\mathcal{G}_n$  with degree possibly at least 3, it follows that  $w = S_n$  and  $S_n$  is in the cycle.

(ii) Suppose that there exists  $D$  adjacent to  $S_n$  which is distinct from  $A_n, B_n, C_n$ . By Lemma 2.4,  $S_n \neq C_n$  and by Lemma 2.6 (i),  $D$  is of degree 1. Since  $S_n$  is adjacent to  $C_n$  by Lemma 2.3 (2), both of the 1-neighborhoods of  $S_n$  are  $\{C_n, S_n, D\}$  (with possibly distinct indices). Thus  $C_n$  and  $D$  belong to the cycle, contradicting that  $D$  is of degree 1.  $\square$

**Lemma 5.2.** *If  $\mathcal{G}_n$  has a cycle not containing  $C_n$ , then  $\mathcal{G}_{n+\ell}$  has a cycle containing  $C_{n+\ell}$  for some  $\ell \geq 1$ .*

*Proof.* By Lemma 5.1 (ii),  $S_n$  is adjacent to  $A_n, B_n, C_n$  only. Since  $S_n$  is the unique vertex of degree at least 3 and it is in the cycle, and since  $C_n$  is adjacent to  $S_n$ , the graph  $\mathcal{G}_n$  is the union of a cycle containing  $S_n, A_n, B_n$  and a line segment containing  $S_n, C_n$ .



Denote the cycle by  $\mathcal{C}_n := [S_n A_n C^1 C^2 \dots C^m B_n S_n]$ . Since  $(n+1)$ -ball extension of  $S_n$  which is adjacent to  $A_n$  (resp.  $B_n$ ) is  $A_{n+1}$  (resp.  $B_{n+1}$ ),  $\mathcal{C}_{n+1} := [S_{n+1} A_{n+1} \overline{A_n C^1} \dots \overline{C^m B_n} S_{n+1}]$  is a cycle in  $\mathcal{G}_{n+1}$ .

Let  $\ell$  be the number of vertices in  $\mathcal{G}_n$  not in the cycle. If  $\ell = 1$ ,  $C_n$  is the unique vertex not in the cycle, thus all vertices in  $\mathcal{G}_{n+1}$  belongs to the cycle  $\mathcal{C}_{n+1}$ .

If  $\ell > 1$ , then  $\overline{D^2}$  is adjacent to every  $\overline{D^1} = \overline{C_n} = S_{n+1}$ , which is followed by  $\overline{D^2} = C_{n+1}$ . Path  $[\overline{D^2 D^3} \dots \overline{D^\ell}]$  in  $\mathcal{G}_{n+1}$  is not in the cycle and  $\overline{D^\ell}$  is degree 1. Repeating this procedure, we obtain the cycle  $\mathcal{C}_{n+\ell}$  in  $\mathcal{G}_{n+\ell}$  which contains all vertices of  $\mathcal{G}_{n+\ell}$ .  $\square$

**Lemma 5.3.** *If  $\mathcal{G}_n$  has a cycle containing  $C_n = S_n$ , then  $\mathcal{G}_{n+1}$  has a cycle containing  $C_{n+1} \neq S_{n+1}$ .*

*Proof.* Suppose that  $\mathcal{G}_n$  has a cycle which contains  $C_n = S_n$ . If  $C_n = S_n$ , then the cycle in  $\mathcal{G}_n$  is  $[S_n A_n C^1 \dots C^m B_n S_n]$ , Either  $A_{n+1} = S_{n+1}$  or  $B_{n+1} = S_{n+1}$ . Thus, by Lemma 2.3 (3), either  $B_{n+1}$  is adjacent to  $S_{n+1} = A_{n+1}$  or  $A_{n+1}$  is adjacent to  $S_{n+1} = B_{n+1}$ . Therefore, we have a cycle either  $[S_{n+1}(= A_{n+1}) \overline{A_n C^1} \dots \overline{C^m B_n} B_{n+1} S_{n+1}(= A_{n+1})]$  or  $[S_{n+1}(= B_{n+1}) A_{n+1} \overline{A_n C^1} \dots \overline{C^m B_n} S_{n+1}(= B_{n+1})]$ . Since  $S_{n+1}$  is adjacent to  $A_{n+1}$ ,  $B_{n+1}$  or  $C_{n+1}$ , either  $\overline{A_n} = C_{n+1}$  or  $\overline{A_n} = C_{n+1}$ . Therefore  $\mathcal{G}_{n+1}$  has a cycle which contains  $C_{n+1} \neq S_{n+1}$ .  $\square$

**Proposition 5.4.** *If  $\mathcal{G}_n$  has a cycle for some  $n$ , then*

- (1)  $\phi$  is of bounded type,
- (2) The graphs  $\mathcal{G}_n^A$  (or  $\mathcal{G}_n^B$ ) for sufficiently large  $n$  are all isomorphic to each other and the quotient graph  $X = \varinjlim \mathcal{G}_n^B$  (resp.  $X = \varinjlim \mathcal{G}_n^A$ ).

*Proof.* By Lemmas 5.2 and 5.3, we may assume that if  $\mathcal{G}_n$  has a cycle containing  $C_n$  which is not  $S_n$ .

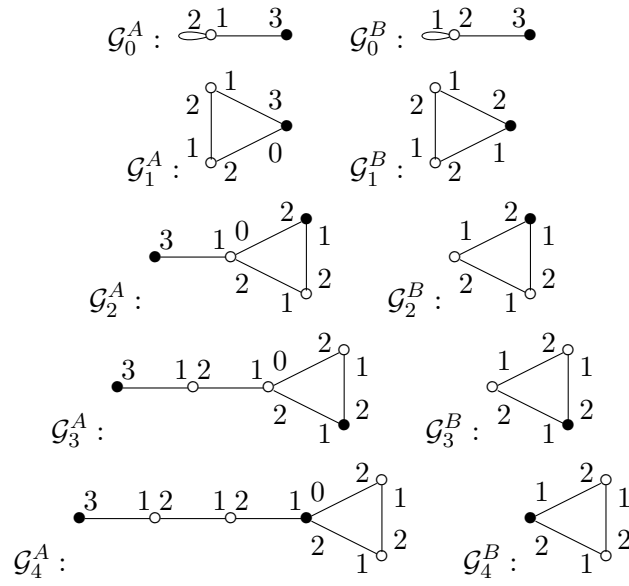
If  $C_n \neq S_n$ , then the cycle in  $\mathcal{G}_n$  is  $[S_n C_n C^1 \dots C^m S_n]$ , where  $C_n, C^1, \dots, C^m$  are distinct. By Lemma 5.1 we have  $C^m = A_n$  or  $B_n$ , say  $C^m = A_n$ . Then  $\mathcal{G}_n^A$  is the edge indexed graph with given by the cycle. Choose an  $n$ -ball  $S_n$  adjacent to  $A_n$  then its extension to  $(n+1)$ -ball is  $A_{n+1}$  which is adjacent to  $S_{n+1} = \overline{C_n}$  and  $\overline{A_n} = \overline{C^m}$ . Therefore,  $[A_{n+1} S_{n+1} \overline{C^1} \dots \overline{A_n} A_{n+1}]$  is the cycle in  $\mathcal{G}_{n+1}$  and  $\overline{C^1} = C_{n+1}$ . By the the same argument for  $\mathcal{G}_n^A$ , we deduce that  $\mathcal{G}_{n+1}^A$  is the edge indexed graph given by the cycle. By Lemma 3.1, we have  $\mathcal{G}_{n+1}^A \cong \mathcal{G}_n^A$ .

By repeating this procedure  $\mathcal{G}_{n+i}^A \cong \mathcal{G}_n^A$  and  $B_{n+i}$  is not a vertex of  $\mathcal{G}_{n+i}^A$  for any  $i \geq 1$ . Let  $x_i$  be a vertex of  $T$  such that  $\mathcal{B}_{n+i}(x_i) = B_{n+i}$  for  $i \geq 1$ . Then  $\mathcal{B}_{n+j}(x_i) \in V\mathcal{G}_{n+j}^B - V\mathcal{G}_{n+j}^A$  for any  $j \geq i$  which implies that  $\mathcal{B}_{n+j}(x_i)$  is not special for any  $j \geq i$ . Thus,  $\phi$  is of bounded type and

each vertex  $x$  in  $T$  satisfies  $\mathcal{B}_{n+i}(x) = B_{n+i}$  for some  $i \geq 1$ . Moreover, the number of neighboring vertices is determined by the edge index of  $\mathcal{G}_n^B$  and the quotient graph  $X = \varinjlim \mathcal{G}_n^B$ .  $\square$

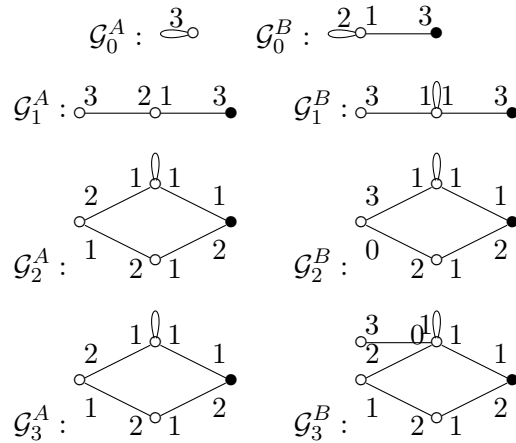
**Example 5.5.** An example of circular graph:

$$X : \bullet \xrightarrow{3} \circ \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \bullet \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \bullet \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \dots$$



**Example 5.6.** An example with an  $n$ -ball  $C$  adjacent to  $S_n$  and which is not one of  $S_n$ ,  $A_n$ ,  $B_n$ ,  $C_n$ . Note that this case can happen only for bounded type Sturmian colorings.

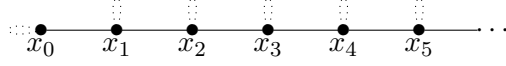
$$X : \circ \xrightarrow{3} \circ \xrightarrow{1\ 0\ 1} \bullet \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \circ \xrightarrow{1\ 0\ 1} \bullet \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \circ \xrightarrow{1\ 2} \circ \xrightarrow{1\ 0\ 1} \bullet \xrightarrow{1\ 2} \dots$$



Let  $n = 1$ . Then any spacial 1-ball is adjacent to the 1-ball with center  $b = \bullet$  which is not a central 1-ball of special  $j$ -balls,  $j = 0, 1, 2$ .

## 6. STURMIAN COLORINGS OF BOUNDED TYPE

Let  $\phi$  be a Sturmian coloring on tree  $T$ . For each vertex  $v$  of  $T$ , we define the *type set* of  $v$  as the subset of nonnegative integer  $m$ 's such that the colored  $m$ -ball centered at  $v$  is special. A Sturmian coloring  $\phi$  on tree  $T$  is said to be *of bounded type* if the type set of each vertex is finite [7]. By Theorem 1.1, we know that the quotient graph  $X$  is an infinite ray. Denote the vertices of  $X$  from the left by  $(x_i)_{i \geq 0}$ :



**Theorem 6.1.** *Let  $\phi$  be a Sturmian coloring.*

- (1) *The coloring  $\phi$  is of bounded type if and only if either  $\mathcal{G}_n^A$ 's or  $\mathcal{G}_n^B$ 's are all isomorphic for sufficiently large  $n$ .*
- (2) *Moreover, if  $\mathcal{G}_n^A$  (resp.  $\mathcal{G}_n^B$ ) are all isomorphic for sufficiently large  $n$ , then the quotient graph  $X = \varinjlim \mathcal{G}_n^B$  (resp.  $X = \varinjlim \mathcal{G}_n^A$ ).*
- (3) *Let  $m$  be the smallest integer such that  $\mathcal{G}_{n+1}^A = \mathcal{G}_n^A$  for all  $n \geq m$ . Then  $\mathcal{B}_{m+i}(x_i)$  is special. In particular,  $\mathcal{B}_m(x_0)$  is the  $m$ -special ball.*

*Proof.* We showed (1) and (2) in Proposition 5.4 for cyclic Sturmian colorings. For (3), by the proof of Proposition 5.4,  $m$  is the smallest  $n$  such that  $\mathcal{G}_n$  has a cycle containing  $C_n$  which is not  $S_n$ . The  $j$ -th vertex from the unique end of  $\mathcal{G}_{n+i}^B$  for  $1 \leq j \leq i$  is the extension of  $B_{n+j}$  which is an extension of  $S_{n+j-1}$ . We need to consider acyclic Sturmian colorings.

If  $\phi$  is of bounded type, then there exists a vertex  $t \in VT$  and some integer  $m$  such that  $t$  is not a center of special ball for all  $n > m$ . For any  $n_k > m + 1$ ,

$$(6.1) \quad \mathcal{B}_{n_k}(t) \in V\mathcal{G}_{n_k}^{\alpha_k} - V\overline{\mathcal{G}_{n_k}^{\alpha_k}}$$

by the first statement of Lemma 3.6. We claim that  $\alpha_{k+l} = \alpha_k$  for all  $l > 0$ . Indeed, otherwise, for the minimal  $l$  such that  $\alpha_{k+l} = \overline{\alpha_{k+l-1}}$ ,  $\mathcal{B}_{n_k+l-1}(t)$  is a vertex in  $\mathcal{G}_{n_k+l-1}^{\alpha_{k+l}}$  thus  $\mathcal{B}_{n_k+l-1}(t)$  is a vertex of  $\overline{\mathcal{G}_{n_k+l-1}^{\alpha_{k+l-1}}}$ , which is a contradiction to (6.1).

Conversely, if  $\mathcal{G}_n^B$  are all isomorphic for sufficiently large  $n$ , then by Lemma 3.6, for any  $\mathcal{B}_n(t) \in V\mathcal{G}_n^A - V\mathcal{G}_n^B$ ,  $\mathcal{B}_m(t)$  is not special for all  $m \geq n$ . Thus  $\phi$  is of bounded type.

For part (2), suppose that there exists  $m$  such that  $\mathcal{G}_{m+\ell}^A$  is isomorphic for all  $\ell \geq 0$ . Choose  $t', t'' \in VX$  of distance larger than  $|V\mathcal{G}_m^A|$ . By Lemma 4.5, for  $n_k > m$ , the  $n_k$ -balls around vertices between  $t', t''$  are in  $\mathcal{G}_{n_k}^{\beta_k}$ , thus  $\beta_k = B$  for  $n_k > m$ .



Part (3) for acyclic colorings follows from Lemma 3.6.  $\square$

**Theorem 6.2.** *Let  $\phi$  be a Sturmian coloring of bounded type, say  $X = \varinjlim \mathcal{G}_n^B$  (the case  $X = \varinjlim \mathcal{G}_n^A$  is similar). Let  $T_0$  be the set of vertices of  $T$  in the class of  $x_0$ . Then for a countable set  $\Lambda$ ,*

$$T - T_0 = \cup_{\lambda \in \Lambda} T_\lambda,$$

where  $\phi|_{T_\lambda} = \tilde{\phi}|_{T_\lambda}$ , for some periodic coloring  $\tilde{\phi}$  with quotient graph isomorphic to  $\varinjlim \mathcal{G}_n^A$ .

Moreover,  $T_0$  is an  $r$ -regular subtree for  $r > 1$ , a disjoint union of edges if  $r = 1$  and a disjoint union of vertices if  $r = 0$ .

*Proof.* Let  $m$  be the smallest integer such that  $\mathcal{G}_{n+1}^A = \mathcal{G}_n^A$  for all  $n \geq m$  as before. Consider the graph  $X$  colored by the  $m$ -balls  $\phi_m$ . By the previous theorem,  $\mathcal{B}_m(x_0)$  is the special ball  $S_m$ . It follows that

$$|\{\mathcal{B}_m(x_i) : i \geq 1\}| = |\{\mathcal{B}_{m+1}(x_i) : i \geq 1\}|,$$

that is, apart from the leftmost vertex  $x_0$ , the rest of the edge-indexed graph gives a periodic coloring on  $T$ .

Let  $r$  be the index of the loop on  $x_0$ . Then the subgraph  $T_0$  consists of  $x_0$  is an  $r$ -regular subtree for  $r > 1$ , a disjoint union of vertices if  $r = 0$  and a disjoint union of edges if  $r = 1$ .  $\square$

**Remark 6.3.** The converse of the theorem does not hold in general. Such a coloring should be a quasi-Sturmian coloring.

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#### REFERENCES

- [1] N. Avni, S. Lim, E. Nevo, *On commensurator growth*, Israel J. of Math. **188** (2012), 259–279.
- [2] H. Bass, A. Lubotzky, *Tree lattices*, Progress in Math., **176**, Birkhauser, 2000.
- [3] M. Burger, S. Mozes, *CAT(−1)-spaces, divergence groups and their commensurators*, J. Amer. Math. Soc. **9** (1996), 57–93.
- [4] J. Cassaigne, *On a conjecture of J. Shallit*, Automata, languages and programming (Bologna, 1997), 693–704, Lecture Notes in Comput. Sci., **1256**, Springer, Berlin, 1997.

- [5] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Edited by V. Berth, S. Ferenczi, C. Mauduit and A. Siegel. Lecture Notes in Mathematics, **1794**. Springer-Verlag, Berlin, 2002.
- [6] G.A. Hedlund and M. Morse, *Symbolic dynamics II: Sturmian trajectories*, Amer. J. Math. **62** (1940) 1–42.
- [7] D.H. Kim and S. Lim, *Subword complexity and Sturmian colorings of trees*, Ergod. Th. Dynam. Sys. Vol **35**, no. 2 (2015) 461–481.
- [8] D.H. Kim and S. Lim, *Hyperbolic tessellation and colorings of trees*, Abstract and Applied Analysis, vol. 2013, ID 706496 (2013).
- [9] A. Lubotzky, S. Mozes, Zimmer, *Superrigidity for the commensurability group of tree lattices*. Comment. Math. Helv. 69 (1994), no. 4, 523–548.
- [10] G. Rauzy, *Mots infinis en arithmetiques* M. Nivat and D. Perrin (Eds) Automata on Infinite Word No. 192 in Lecture Notes Comp. Sci. pp165-171 (1985), Springer Verlag.
- [11] C. Series, *The geometry of Markoff numbers*, Math. Intell. **7** (1958), 20–29.
- [12] J.-P. Serre, *Trees*, Translated from the French by John Stillwell, Springer-Verlag, 1980.

DEPARTMENT OF MATHEMATICS EDUCATION, DONGGUK UNIVERSITY, PILDONG 1 GIL, JONGRO-GU, SEOUL, 04620

*E-mail address:* kim2010@dongguk.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, KWANAK-RO 1, KWANAK-GU, SEOUL 08826

*E-mail address:* slim@snu.ac.kr